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Investigation of Prospective Mathematics Teachers' Proof Completion Processes Supported By Key Ideas[*]

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Abstract: The aim of this research was to examine prospective mathematics teachers' completion processes of mathematical proofs supported by key ideas in the field of abstract algebra. Prospective teachers were asked to complete the parts of a proof left incomplete using those key ideas. It was decided, together with three academics who are experts in the field of algebra, which theorems to use and which proof sections to leave missing in the proof completion form. The proof completion form consisted of five proofs and was applied to five participants. The participants' processes for completing the proofs were evaluated in semi-structured interviews supported by the think-aloud method. Interviews were recorded on video and transcribed, and descriptive analysis was performed. The findings obtained were analyzed with think-aloud protocols. According to the results, key ideas can be said to play an active role in the proof process and the teaching of proof. In addition, developing activities consisting of proofs supported by key ideas can contribute to prospective teachers' abilities to prove without memorization and internalize proof processes.

Keywords: Key idea, mathematical proof process, teaching of proof, think-aloud, prospective mathematics teachers

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Introduction

Mathematics is the process of revealing, determining, and logically proving properties specific to abstract objects such as numbers, points, and set functions (Yildirim, 2015). When mathematicians face any problem situations, after examining the special cases, they reach generalizations, make assumptions, and prove those assumptions (Arslan & Yildiz, 2010). A mathematician is proud of the fact that mathematics is a more "real" science than any other science, and that things proved in mathematics are never wasted (Cajori, 2014).

Proof has an important role in the development of mathematical thinking skills and the formation of mathematical knowledge among students (Dede & Karakus, 2014). However, students get into a deadlock and often fail because they don't know what to do when creating a proof. In order to understand the causes of this failure, it is necessary to examine students' processes of creating proofs (Weber, 2001). According to Stewart and Thomas (2019), the reason why students have many difficulties proving is that each component in a proof is packed with different conceptual ideas. Therefore, it is unrealistic to expect students to logically combine many definitions, results, and other theorems that will help complete the proof.

Many students do not like proving very much because proofs have a hierarchical structure, and sometimes there are subproofs and subconstructions within proofs. Students' attempts to demonstrate without using this hierarchical structure will fail. It is necessary to tell students that proofs are not generally conceived of in the order they are written. Students need to be encouraged to write parts of a tentative proof 'out of order' (Selden & Selden, 2008). A key idea is an important part of the proof, which gives a sense of why a claim is true and directs the proof (Raman, 2002). A complex proof can be broken down with the help of key ideas and completed in a hierarchical order. The proofs used in this research are similarly broken down with the help of key ideas. Each part of the proof process serves as a clue to the previous and next steps. Therefore, each key idea or step of the proof is also a clue for the preceding and following proof steps.

Mathematical proof and proving are considered important mathematical activities by mathematicians and mathematics teachers. Therefore, every mathematics student at the university level should understand and be able to produce mathematical proofs (Basturk, 2010). Prospective mathematics teachers face mathematical proofs in many courses during their education. According to Gallagher and Infante (2021), courses such as calculus, linear algebra, abstract algebra, and real analysis are the requirements for the completion of undergraduate level mathematics in general. Abstract algebra is full of definitions and theorems that all require proof, and students need to understand every definition and theorem they learn and be able to organize the concepts needed to prove theorems (Agustyaningrum et al., 2020). Considering that prospective teachers learn hundreds of theorems and proofs during their education, it can be thought that the learned knowledge is largely acquired based on memorization (Morali, Ugurel, Turnuklu, & Yesildere, 2006). For this reason, prospective mathematics teachers should be trained in terms of mathematical proof (Guler & Dikici, 2012). In this context, it is necessary to examine the processes of prospective mathematics teachers in writing predetermined key ideas in mathematical proofs.

The aim of this research is to examine pre-service mathematics teachers' processes of completing proofs in the field of abstract algebra, which were divided into parts with the help of key ideas. The pre-service teachers were asked to complete the missing key ideas using the key ideas given as clues and to express their thoughts aloud. In this way, it was also aimed to understand their processes of proving and their thoughts in those processes, and therefore to understand what they could not do, why they could not do it, and how they used key ideas. It is thought that this research will provide instructors with better ideas about students' abilities to prove and use key ideas and thus contribute to the proof-teaching process and future studies.

The relation between mathematical proof and the key idea

To construct a proof requires the accumulation of complex skills consisting of content knowledge, belief in being able to proof, and problem solving ability (Selden & Selden, 2008). For mathematicians, most of the proving process happens in the mind (Raman, Sandefur, Birky, Campbell & Somers, 2009). Key ideas give us a sense of understanding a proof and being convinced. The key idea is the idea that shows why a particular claim is true, creates a sense of understanding the proof (Raman, 2003), and some proofs have more than one key idea (Raman et al., 2009). A key idea of proof could be any mathematical idea, strategy, specific approach, or proving technique that one needs to keep in mind in order to make proof (Yan, 2019).

While proving a theorem, it is recommended to start from the conclusion and move towards the premises while keeping in mind that the proof must proceed in the opposite direction, from the premises to the conclusion. Keeping these two processes and their logical relations in mind at the same time forces the limited resources of memory, which students may have difficulty proving (Koichu & Leron, 2015). Because the students determine their approaches for the proof by trying to comprehend the rationale of the given statement and utilizing their own knowledge (Guler, 2013), students are asked to think about the structure of the proof before starting the proof, and if statements or definitions are explained to help them start the proof after this thinking process, it can be possible to prevent the difficulties they face in proving. For instance, it would be beneficial for the students, who are trying to prove the theorem "f and g being two functions on A, if $f\ o\ g$ is one-to-one then g has to be one-to-one, too.", to explain the meaning of g being one-to-one. In this way, the proof could be started by writing "let $g(x) = g(y)$, x and y being to elements in the domain of g; this start explains that the intended result is "$x = y$". Here, it is seen that "if $g(x) = g(y)$ then $x = y$". Then the students would focus on how they could make use of the hypothesis that $f\ o\ g$ is one-to-one (Selden & Selden, 2008). The definition of g as one-to-one, which is expected to help prove the theorem in the above example, also gives a sense of verification of the theorem and serves as a hint, i.e., it is the key idea. Accordingly, key ideas are expected to be a supportive factor in the completion of proofs.

Examining the thoughts of the individuals involved in the process is thought to be possibly useful in terms of being able to determine the effect of key ideas on the completion process of proofs and revealing the proof process in detail. In such a case, individuals are required to verbalize their thoughts. In this research, the think-aloud method was taken as the basis at certain stages of individuals' processes of verbalizing their thoughts.

The Relationship between the Think Aloud Method and Mathematical Proof

Think aloud protocols are becoming more common with each passing day in educational research because of the richness of data that can possibly be derived from the methodology (Johnstone, Bottsford Miller & Thompson, 2006). It is possible to make an objective observation about students' behaviors and mental schemes by using the think aloud method (Kayacan, 2005). Mathematical proving is a complex process involving many intellectual stages (Cetin & Dikici, 2016). It is necessary to analyze proof processes and intellectual stages in mathematical proofs to know where and for what reasons students make mistakes. According to Cetin and Dikici (2016), it is necessary to clearly identify what the individual thinks, what he/she makes and why he/she makes it, and what information he/she uses when creating proof stages. Therefore, it is necessary that the opportunity to analyze the intellectual processes of individuals in detail be provided. During the use of the think aloud method in mathematical proving and problem solving, the individual is asked to express his/her thoughts verbally, i.e., verbalize his/her thoughts in the process, with the methods and techniques he/she uses. According to Gerard Scallon (2004), think aloud is a verbalization process that makes problem solving processes easier. Individuals' expression of their thoughts out loudly during problem-solving contributes both to the solution process and to achieving the correct result, and it also helps reveal the ways and methods that students use in the activity (cited from Gunes, 2012). Therefore, mistakes in students' problem-solving processes can also be identified through the think aloud method.

According to many researchers, proof construction is a problem-solving task, and the process of proving and the process of problem solving are two intertwined processes (Koichu & Leron, 2015; Mamona-Downs & Downs, 2005; Palatnik & Dreyfus, 2019; Weber, 2001). According to Weber (2001), proving is a problem solving activity. In problem solving, the individual who will solve the problem is offered an initial statement and is asked to take a series of actions that will transform the initial statement into the intended target statement. When creating the proof of a statement, the individual creating the proof is offered a series of assumptions and is asked to derive a series of inferences that will lead to the statement, which will be proved. According to Guler (2013), considering the relationship between problem solving and proving, it is seen that problem solving and proving processes are complementary and intertwined, and that the difficulties faced in problem solving processes in algebra also affect the proof approaches.

Considering all these points, it can be said that the think-aloud approach facilitates not only problem-solving processes but also proof processes, and it also helps to clarify the approaches and methods used during the proof stage, the difficulties faced, and the mistakes made.

Method

The case study design, one of the qualitative research methods, was used in the study. Case studies have three attributes: focusing on a specific event, program, or phenomenon; describing the resultant product and the phenomenon investigated in a rich way and enabling the readers to better understand the phenomenon in the study and confirm their knowledge (Merriam, 2009/2013). Therefore, in this research, the processes of prospective mathematics teachers' writing pre-determined key ideas were focused on, and in this process, the findings obtained as a result of the research were richly described. In addition, it was ensured that the reader could better understand and experience the mathematical proof steps with the help of both the process and key ideas.

Participants

In order to enhance the practicality of the think aloud method, the participants should be selected from individuals whose verbal expression skills and knowledge levels are advanced in the field of learning that constitutes the subject of the study (van Someren, Barnard & Sandberg, 1994). Although thinking aloud is a simple and easy-to-implement method, it may become ineffectivein some cases. For some individuals, it is difficult to track, verbalize, and express thoughts. Some students have difficulty expressing their thoughts and abstain from speaking since they feel ashamed of teachers or other individuals or due to the fear of humiliation. Some others, on the other hand, get confused by the method and have difficulty defining its intellectual processes and stages (Gunes, 2012). When selecting participants, attention should be paid to minimizing such potential effects (van Someren et al., 1994). In this research, among 15 prospective teachers who voluntarily attended the pre-implementation training, individuals were selected who know themselves, can clearly express their intellectual processes, can adapt to the method of thinking aloud and can communicate with the researcher smoothly. The researchers were able to decide whether or not prospective teachers have the characteristics suitable for implementation with the help of their informal observations and witnessing the education process of the prospective teachers as well as their attitudes and behaviors in that process for a period of three years.

Another point to take into consideration in determining the participants in the think aloud method is to ensure that the participants have knowledge about the algebraic subjects that will be used in the implementations. For this reason, the prospective teachers who would participate in this research were selected on a voluntary basis among the prospective teachers who attended the pre-implementation training; can clearly express themselves; can communicate with the researcher smoothly; had previously taken an Abstract Algebra course, and therefore can provide rich data.

Due to the labor-intensive nature of this method, the sample size involved in the study is small. However, small numbers do not necessarily mean small data sets. Since the study process is intensive, small sample sizes can still provide reliable information. (Johnstone et al., 2006).

Considering all these reasons, the research was carried out with a group of five prospective teachers consisting of two boys and three girls from a state university's department of mathematics education who had completed their third grade education and volunteered to participate in the research process.

Data Collection Tools

A semi-structured interview supported by the think aloud method

Semi-structured interviews combine fixed alternative answers and the ability to investigate deeply in the relevant field; and have some advantages such as enabling the interviewee to express himself/herself and gain in-depth information when necessary (Buyukozturk, Kilic Cakmak, Akgun, Karadeniz & Demirel, 2010). In semi-structured interviews, the researcher prepares the questions beforehand, but allows for the rearrangement and discussion of the questions by providing partial flexibility to the participants during the interview; therefore, participants have control over the study, as well (Ekiz, 2009). The researcher also gains the opportunity to learn in depth about the opinions of the participants about the study’s subject (Guler, Halicioglu & Tasgin, 2013). In this research, participants were expected to complete the parts of the proof left incomplete, by using the key ideas. In the proof completion form, it was decided, together with three academicians who are experts in the field of algebra, which theorems to use and which proof sections to leave missing. Therefore, a pre-prepared proof completion form consisting of the proofs of five theorems was given to the participants, and the processes of completion of the proofs by the participants were tried to be revealed through semi-structured interviews. However, in the thinking process intended for proof, which consists of many intertwined processes, these semi-structured interviews were tried to be supported through the think aloud method in order to reveal the participants’ thoughts in a depth-oriented way and to reveal their competence to use the given key ideas. When the participants got away from thinking aloud, the researcher tried to keep the interviews proceeding with the think aloud method by using various expressions (‘What do you think? Tell me out loud what you are thinking. What do you have in your mind? What did you try to say, what did you understand?). When the participants had difficulty describing their intellectual processes, the researcher guided the process with various sentences that he composed for the proof (Why should we show it? What features did we use there? What did he generally do in this proof? How can we combine an upper and a lower digit?). In cases where no problem was encountered in thinking aloud, the researcher did not intervene in the process, and provided the prospective teacher with the opportunity to express himself. The researcher also tried to enable the prospective teachers to express their thoughts verbally, by asking questions such as. "What helped you here? What did he generally do in this proof? Why did you write like this?

The preparation process of the proof completion form supported by key ideas

First of all, after the researchers informed three academicians who are experts in algebra about the purpose of this research, they asked the experts to convey their opinions on whether the predetermined seven abstract algebra theorems were suitable for this purpose. In the proof completion form to be prepared, the researchers got ideas from experts about which theorems should be used and which key ideas (hints) should be given. Three theorems, on which the experts agree that they are more suitable for the cognitive levels of the prospective teachers, have been selected for use in this research. In addition to these three theorems, one of the experts stated that, due to his experiences, a theorem for the proof of homomorphism should be included and the process step for using the property of homomorphism should be left incomplete. Another expert stated that the proof of the "$(G,.)$ be a group and $H \leq G$. For $\forall\, a \in G$, $Ha = \{x \in G : a \equiv x (mod\, H)\}$, i.e $Ha = \bar{a}$." theorem, which he thought was appropriate for the cognitive levels of the prospective teachers, should be included in the practice. In light of these opinions, it was decided that five theorems would be used in the proof completion form and that key ideas would be given. It was also decided with these experts which key ideas to give and which steps/key ideas to leave incomplete in the determined proofs. As a result, in the proofs of the 5 theorems in the proof completion form, some key ideas were left incomplete and others were given as hints. Prospective teachers were asked to complete the missing key ideas using the key ideas given.

The Implementation Process

During the pilot implementation process, problems occurred because the participants were unfamiliar with the think aloud method and were silent from time to time. Before the main application, two activities were organized for the volunteers, the first of which was 40 minutes and the second was 15 minutes. In the first of the activities attended by 15 volunteer pre-service mathematics teachers, what thinking aloud really is, its benefits, principles, criteria, and its effect on improving problem solving skills were explained, and a sample theorem proof was made with the method of thinking aloud. Until the second activity, a theorem that the participants wouldprove individually using the method of thinking aloud was given as homework.

In the second activity, two weeks after the first activity, the given homework was proven with the participants' method of thinking aloud. The opinions of the participants about the method of thinking aloud were taken informally. In this informal interview, it was tried to determine whether the prospective teachers perceived the method of thinking aloud at the end of the activities and whether they internalized the method until the second activity.

During the pilot implementation process, it was observed that some of the participants had difficulty expressing their opinions and were ashamed of the researcher or they avoided speaking because they were embarrassed that the proof completion processes were being followed by someone. Therefore, at the stage of determining the participants in the main implementation, 15 volunteer prospective mathematics teachers who attended the pre-implementation activities had been selected by paying attention to minimizing these possible effects; Practices had been carried out with five teacher candidates who could express themselves clearly, adapt to the think aloud method, and clearly communicate with the researcher. The processes of prospective teachers to complete the proof completion form supported with key ideas and thinking aloud had been recorded with a video recorder.

Data Analysis

In the thinking process, successive logical operations are done in our mind. For example, problem solving, decision making, critical thinking, reasoning, creative thinking, etc. These appear when thinking aloud in the forms of words, sentences, speeches, beliefs, suggestions, judgment, intellectual images, descriptions, methods, and techniques (Gunes, 2012). Therefore, analyzing a think aloud protocol is considerably more difficult than directly analyzing an audio recording. When analyzing think aloud protocols, transcripts are divided into pieces and turned into a standard form (van Someren et al., 1994).

In this research, descriptive analysis was performed on the semi-structured interviews supported by the think aloud method. In descriptive analysis, the data is systematically and clearly described, and then these descriptions are interpreted, and the cause and effect relationships are examined to reach certain conclusions (Yildirim & Simsek, 2008). Before performing the analyses, the video recordings created during the interviews were transferred to the computer environment; and the proof completion operations were transcribed through the audio and video transcripts. The staging process was carried out in accordance with the think aloud protocols, by viewing the video recordings four times at different times, and reading the transcripts three times, and then the stages were presented in tables. The comments defining the intellectual processes of the prospective teachers were revealed by evaluating the video recordings and the proof completion processes together.

Data analysis was based on the sample protocol analysis of the think aloud method for the problem solving process given by van Someren et al. (1994). In the think aloud method, spoken expressions are referred to as verbal protocol, while written ones are referred to as written protocol (van Someren et al., 1994). In this research, both the verbal and written protocols were delivered to the reader simultaneously, and by doing so, it was aimed at analyzing the opinions of the prospective teachers in the proof completion process, with a holistic approach. In the research process, a total of 25 theorem proofs were examined with protocol analysis; however, a sample protocol analysis intended for each theorem was presented for delivering the findings to the reader. This way, it is tried to simplify the research data.

Validity, Reliability and Ethical Issues

The transferability of the research was increased by determining that the participants who attended the pre-implementation training can clearly express themselves, communicate with the researcher smoothly, have previously taken an Abstract Algebra course, and therefore can provide rich data. In addition, the processes of data collection and analysis, the application process has been conveyed to the reader in a detailed explanation. The resulting findings had been presented with integrity through direct transfer. In order to ensure the consistency of the research, the participants had not been treated differently in the semi-structured interviews, and the process had been recorded.

The thinking aloud method prevents researchers from commenting and creates an objective method by making verbal protocols accessible to everyone as data (van Someren et al., 1994). Therefore, in this research, the data was coded sentence by sentence based on the thinking aloud method.

In this research, all the participants were informed about the purpose of the research. The names used are the nicknames given to the participants by the researchers. A volunteeri agreement that informs the participants about the research has been read and signed by the participants. This volunteering agreement is also a contract that guarantees the participants that the interviews will be video-recorded, but their personal information will be protected, and that the participants who have decided to take part and leave later will not experience any negativity.

Findings

In this section, the sample findings intended for proving the five theorems that examine the proof completion process in abstract algebra were given by doing the required operations with the help of the key ideas. It has been observed that the prospective teachers effectively used the key ideas when completing the sections left incomplete, but that Umut tried to make explanations in instead of benefiting from the key ideas in some theorems, and consequently he could not use the key ideas in such a way as to make them helpful in the proof. The researcher warned the prospective teachers when they did not notice the key ideas or when they omitted to think aloud due to focusing on the flow of the proof.

To begin the presentation of the results, an overview of the full set of 25 analyses was shared. An example of the pre-service teachers' solutions was then presented for each theorem. The proof of the relevant theorem was also included in the sample analyses that were presented. Steps left incomplete in the proof completion form that were expected to be completed by pre-service teachers were highlighted in red.

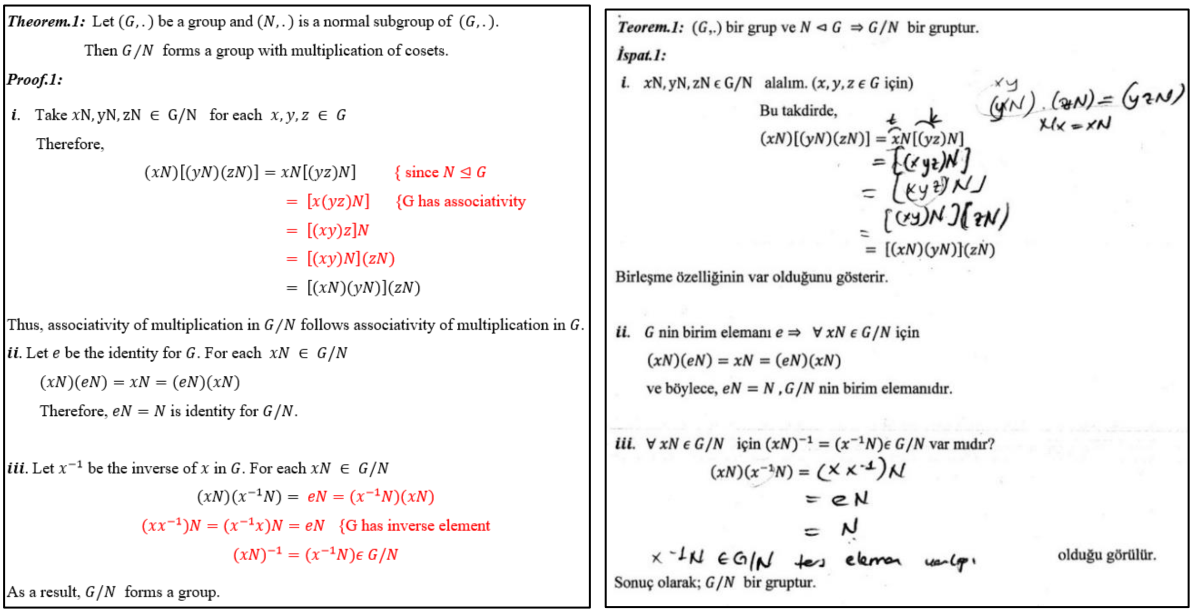
Findings Regarding the 1st Theorem

In the first step of the 1st theorem, where the existence of the associative property was investigated using the normal subgroup definition, most of the prospective teachers did not use the normal subgroup definition, and they tried to complete the proof by basing their operations on the key ideas given at the beginning and end of the steps left incomplete. At this point, although Hasan alone expressed the definition of a normal subgroup, he tried to correlate the key ideas given at the beginning and end to complete the part that was incomplete but advanced its operations by basing them on the key idea given in the last step.

In the third step of the 1st Theorem, the prospective teachers were expected to complete the proof in accordance with the conceptual information given as a key idea in the second step, and Umut alone could complete the third step completely since he elaborated on the conceptual information in the second step in detail. The other prospective teachers could precede their operations only unilaterally, and consequently, they could not complete the proof at the expected level, since they based their operations on the equality given as the key idea at the beginning of the third step. In addition, although Ceyda used the key ideas, she failed to complete the proof because she perceived the operations, based on the definition of the normal subgroup, as distributive property.

The proof of Theorem 1 that examines the processes of pre-service teachers to continue the proof by performing the necessary operations using key ideas, and the findings of Hasan's proof completion process are presented below.

Video recordings of Hasan's solutions regarding the proof of the 1st theorem given in the proof completion form, the transcript of the video recordings, and the solutions in the form were examined together, and the following comments were made.



Theorem.1: Let $(G,.)$ be a group and $(N,.)$ is a normal subgroup of $(G,.)$.
Then G/N forms a group with multiplication of cosets.

Proof.1:

i. Take $xN, yN, zN \in G/N$ for each $x, y, z \in G$

Therefore,

$(xN)[(yN)(zN)] = xN[(yz)N]$ {since $N \trianglelefteq G$

$= [x(yz)N]$ {G has associativity

$= [(xy)z]N$

$= [(xy)N](zN)$

$= [(xN)(yN)](zN)$

Thus, associativity of multiplication in G/N follows associativity of multiplication in G.

ii. Let e be the identity for G. For each $xN \in G/N$

$(xN)(eN) = xN = (eN)(xN)$

Therefore, $eN = N$ is identity for G/N.

iii. Let x^{-1} be the inverse of x in G. For each $xN \in G/N$

$(xN)(x^{-1}N) = eN = (x^{-1}N)(xN)$

$(xx^{-1})N = (x^{-1}x)N = eN$ {G has inverse element

$(xN)^{-1} = (x^{-1}N) \in G/N$

As a result, G/N forms a group.

Teorem.1: $(G,.)$ bir grup ve $N \lhd G \Rightarrow G/N$ bir gruptur.

İspat.1:

i. $xN, yN, zN \in G/N$ alalım. ($x, y, z \in G$ için)

Bu takdirde,

$(xN)[(yN)(zN)] = xN[(yz)N]$

$= [(xyz)N]$

$= [(xy)N][(zN)]$

$= [(xN)(yN)](zN)$

Birleşme özelliğinin var olduğunu gösterir.

ii. G nin birim elemanı $e \Rightarrow \forall xN \in G/N$ için

$(xN)(eN) = xN = (eN)(xN)$

ve böylece, $eN = N$, G/N nin birim elemanıdır.

iii. $\forall xN \in G/N$ için $(xN)^{-1} = (x^{-1}N) \in G/N$ var mıdır?

$(xN)(x^{-1}N) = (xx^{-1})N$

$= eN$

$= N$

olduğu görülür.

Sonuç olarak; G/N bir gruptur.

Figure 1. Proof of the 1st Theorem Given in the Proof Completion Form and Hasan's Solution

A sample protocol analysis is given below in Table 1, where data on the pre-service teachers' abilities to use key ideas are interpreted. Pre-service teachers' think-aloud processes for the proof of each theorem were interpreted with the help of these protocols.

Table 1.

Interpreting the Data Regarding Hasan's Ability to Use the Key Ideas in the 1st Theorem

Pre-service Teacher's Sentence	Comment
So, we need to make changes to get here [indicating the key idea '$[(xN)(yN)](zN)$' given in the last step].	He is interpreting the key idea $[(xN)(yN)](zN)$ given in the last step.
I guess there was a normal subgroup transformation. It was like this. It must have had a transformation like $(yN).(zN) = (yzN)$. That's why we did it that way. How do we do this? … If it is a normal subgroup, it has an equality of form $Nx = xN$. Maybe we can use it again here. We can write it as $= Nx.(yz)N$.	He remembers his previous knowledge of the definition and characteristics of the normal subgroup and, based on these, completes the second step with an incorrect statement.
[Upon the interviewer's warning] So, we want to remove zN. You know, we show associativity. How exactly can we show this? This is what we would do if we went back. For $[(xy)N]$, we had (zN) here [based on the last step, he wrote '$[(xy)N](zN)$'	Upon the interviewer's warning, he reviews the key idea given in the last step and starts to perform the procedures towards the proof, starting from the last step.

I got it. We're going to liken it to this [by showing the key idea '$(xN)[(yN)(zN)]$' given in the first step], so it's like this. We will probably write it as $[(xyz)N]$. So, when we go backwards, it will become like this. The only place left is there [deleting the second step where it was previously written '$Nx.(yz)N$'].	He connects the key idea '$(xN)[(yN)(zN)) = xN[(yz)N$' given in the first step with the key idea'= $[(xN)(yN)](zN)$' given in the last step and starting from the last step, he completes the next step with the right operations.
How are we going to do this? Hmm, we can do that in a similar way. So, taking this (xN) example as t, and this (yz) as k, it will be tkN, as if I had only one N. t is already x, and the other is yz, then it will be $[(xyz)N]$. So, we have got the same here.	Although he did not express the steps $(x(yz))N = ((xy)z)N$, he was able to partially complete the proof.
This is what helped me. The feature here (also showing the ['$(yN)(zN) = (yzN)$ he wrote] So, the feature I just wrote. Both this and going back here [showing the key idea, the '$[(xN((yN](zN)$' given in the last step] helped a little.	He states that he got help from the key idea given in the last step and the definition of the normal subgroup.
[After reading the step in which the identity element's existence was searched for] Now, it is trying to display the inverse element. You know, to show it is a group. There is an identity element. We could write it like this. We can write it as $(x.x^{-1})N = eN$. This is also equal to N. We already found N above, it was the identity element. If E goes to my element, $(x^{-1}N)$ will be inverted. So, it will be the inverse element. Here we can write $(x^{-1})N$ element G/N. There, the existence of the inverse element.	He knows the inverse element property but does not express the existence of the inverse element in the G group. He writes $(xN)(x^{-1}N)$ as $(xx^{-1})N$ with the help of the key ideas given in the steps i. and ii.

As seen in Table 1, while considering the existence of associativity, Hasan interpreted the key idea given in the last step before doing the operations, tried to establish a connection between the key ideas given at the beginning and end of the relevant part in order to complete the missing part by expressing the normal subgroup definition and properties, and advanced his operations based on the key idea given in the last step. Although he associated key ideas and expressed the normal subgroup definition, he partially completed the proof by misrepresenting some operation steps. While investigating the existence of the inverse element, he brought the proof to a certain level based on the key ideas given in the identity element and inverse element features and the statement he wrote about the normal subgroup definition, although he did not express it verbally. Since he missed the key idea that would help him choose and use the conceptual information, he performed the operations unilaterally, so he could not complete the proof at the expected level.

Findings Regarding the 2nd Theorem

In the process of completing the proof of the 2nd theorem, all the prospective teachers effectively utilized the key ideas and they all tried to develop a solution based on the key ideas in all steps. They also corrected their mistakes with the help of key ideas again, while completing the steps left incomplete. In completing the step that was expected to be completed using previous knowledge, the prospective teachers tried to correlate the key ideas given at the beginning and end of the step left incomplete instead of using the previously given information. By doing so, they could complete the proof, but not exactly. Unlike other prospective teachers, Umut did not know how to complete the step left incomplete and failed in completing it because he based his operation on the key idea given in the previous step, instead of correlating the previous and next steps.

As an example of the analysis of the 2nd theorem, findings and comments on Gul's proof process are presented below.

Theorem.2: Let $(G,.)$ be a group and $(H,.)$ is a subgroup of $(G,.)$. Then,

$Ha = \{x \in G : a \equiv x(mod\ H)\}$ for each $a \in G$. That is $Ha = \bar{a}$, where Ha is a right coset and $\bar{a}$ is an equivalence classes of a.

Proof.2: In order to prove that $Ha = \bar{a}$ for every $H \leq G$ we must show that $\bar{a} \subset Ha$ and $Ha \subset \bar{a}$.

i. Let us suppose that $x \in \bar{a}$. In this case $a \equiv x(mod\ H)$ and then $ax^{-1} \in H$ from the definition of the equivalences of two elements according to modulo a subgroup H. Since $H \leq G$ there exist $h \in H$ such that $ax^{-1} = h$. Multiply both sides of this equality by a^{-1} from the left hand side to obtain

$a^{-1}(ax^{-1}) = a^{-1}h$ (use of associativity)

$(a^{-1}a)x^{-1} = a^{-1}h$ (use of the inverse element)

$ex^{-1} = a^{-1}h$ (use of the identity element)

$x^{-1} = a^{-1}h$. Take the inverse both sides of the equality to get

$(x^{-1})^{-1} = (a^{-1}h)^{-1}$

$x = h^{-1}a$.

(Since $h \in H$ and $H \leq G$ then $h^{-1} \in H$ and recall $h^{-1} = h_0 \in H$) to get

$x = h_0 a \in Ha$

Thus $\bar{a} \subset Ha$.

ii. Take $y \in Ha$. Then there exist $x \in H$ such that $y = xa$. Multiply both sides of this equality by a^{-1} from the right hand side to obtain

$ya^{-1} = (xa)a^{-1}$ (use of associativity)

$ya^{-1} = x(aa^{-1})$ (use of the inverse element)

$ya^{-1} = xe$ (use of the identity element)

$ya^{-1} = x$ Therefore we find that $x = ya^{-1} \in H$. Thus

$ya^{-1} \in H \Longrightarrow y \equiv a(mod\ H) \vee a \equiv y(mod\ H)$

that shows $y \in \bar{a}$ which means that $Ha \subset \bar{a}$ and the proof completes.

Teorem.2: $(G,.)$ bir grup ve $H \leq G$ olsun. $\forall a \in G$ için,

$Ha = \{x \in G : a \equiv x(modH)\}$ yani $Ha = \bar{a}$ dir.

İspat.2: $a \equiv x\ (modH)$ kongrüansına uyan $x \in G$ lerin kümesini $\bar{a}$ ile gösterelim. Yani $\bar{a}$, a nın denklik sınıfı olsun. $H \leq G$ için $Ha = \bar{a}$ olduğunu dolayısıyla $\bar{a} \subset Ha$ ve $Ha \subset \bar{a}$ olduğunu göstermeliyiz.

i) $x \in \bar{a}$, yani $a \equiv x(modH) \Rightarrow ax^{-1} \in H$ yazılır. H bir altgrup olduğundan kapalılık özelliğine göre öyle bir $h \in H$ vardır ki; $ax^{-1} = h$ dir. Buradan;

Soldan a^{-1}/ $ax^{-1} = h$ .... (Grupta birleşme özelliğinden)

$a^{-1}ax^{-1} = a^{-1}h$ .... (Grupta ters eleman özelliğinden)

$ex^{-1} = a^{-1}h$ (Grupta etkisiz eleman özelliğinden)

$x^{-1} = a^{-1}h$ olur. Eşitliğin her iki yanının tersi alınırsa

$(x^{-1})^{-1} = (a^{-1}h)^{-1}$

$x = h^{-1}a$ olur.

($h \in H$ ve $H \leq G$ olduğundan $h^{-1} = h_0$ ....denirse)

$x = h_0 a \in Ha$ olur ki buradan $\bar{a} \subset Ha$ bulunur.........(1)

ii) Tersine olarak $y \in Ha$ alınırsa; $x \in H$ vardır öyle ki; $y = xa$ dır. Buradan;

$y = xa$ / a^{-1} sağdan (Grupta birleşme özelliğinden)

$ya^{-1} = xaa^{-1}$ (Grupta ters eleman özelliğinden)

$ya^{-1} = xe$ (Grupta etkisiz eleman özelliğinden)

$ya^{-1} = x \in G$ olur. Buradan $x = ya^{-1} \in H$ bulunur.

$ya^{-1} \in H \Rightarrow y \equiv a(modH) \vee a \equiv y(modH) \Rightarrow y \in \bar{a}$ bulunur ki;

$Ha \subset \bar{a}$ demektir..............(2)

(1) Ve (2) den $Ha = \bar{a}$ denir.

Figure 2. Proof of the 2nd Theorem Given in the Proof Completion Form and Gul's Solution

Gul failed to complete the first steps due to her inability to use the associative property and her lack of mathematical notation knowledge, even though she actively used the key ideas and knew in what manner she needed to do her operations. She considered the key ideas as a whole, which were given before and after the steps left incomplete, and by doing so, she tried to decide what path she would follow. Although she was able to completely write the identity element and inverse element concepts in the relevant spaces without hesitation in the proving process, she hesitated while completing the same steps in the second section, and was able to complete those steps after being sure by reviewing her operations. That situation is thought to result from memorization or the habits formed by always showing the group characteristics in the same order. She was able to complete the expression 'because $h \in H$ and $H \leq G$ if .... is said that' which she found the most difficult, by correlating the key ideas given before and after the statement, but not exactly, due to omitting to define $h^{-1}, h_0 \in H$. Consequently, she could not write the statement regarding the existence of the inverse element in the group.

Findings Regarding the 3rd Theorem

In the proof of the 3rd theorem, the prospective teachers were asked to determine the hypothesis and judgment, and then the steps following the hypothesis and judgment were given as the key idea. The prospective teachers benefited from the key ideas in determining the hypothesis and judgment. Ceyda, who tried to determine the hypothesis and the judgment based on the expression of the theorem, edited the expressions, which were incorrect in terms of meaning and mathematical representation, with the help of the key ideas again, while Hasan did not edit his statement, although he read the key ideas. In the 3rd Theorem, Umut, unlike other prospective teachers, could not use the key ideas effectively and consequently failed in determining the hypothesis and judgment, and he also could not express these appropriately.

Below are findings and comments on Ceyda's proof process as an example of the analysis of Theorem 3, which examines the process of starting the proof with the help of key ideas.

Theorem.3: The intersection of finite (or infinite) number of subgroups of a group $(G,.)$ is a subgroup. ***Proof.3:*** Let us consider the set $I = \{1,2,3,...\}$. Let us suppose that H_i $(i \in I)$ is the subgroup of $(G,.)$, i.e. $(H_i \leq G)$. Hence, we state $H = \bigcap_{i \in I} H_i$. So we must show that $H \leq G$. Take $a, b \in H$ for this. → Using the definition of intersection, $a, b \in H_i$ for $\forall\, i \in I$ → Using the definition of subgroup, $a.b^{-1} \in H_i$ for $\forall\, i \in I$ → Using the definition of intersection, $a.b^{-1} \in H$. Finally we get $H \leq G$.	***Teorem.3:*** Bir $(G,\cdot)$ grubunun birtakım (sonlu veya sonsuz) altgruplarının kesişimi yine G nin bir altgrubudur. ***İspat.3:*** $I = \{1,2,3,...\}$ bir indis kümesi olsun. ....$\bigcap$Hi = H almak üzere .....H ≤ G.......... .... olduğunu göstereceğiz. Bunun için $a, b \in H$ alalım. → Kesişimin tanımından; $\forall\, i \in I$ için $a \cdot b \in H_i$. → Altgrubun tanımından; $\forall\, i \in I$ için $a \cdot b^{-1} \in H_i$ → Kesişimin tanımından; $a \cdot b^{-1} \in H$ olur. Öyleyse $H \leq G$ dir.

Figure 3. Proof of the 3rd Theorem Given in the Proof Completion Form and Ceyda's Solution

Although Ceyda constructed an incorrect hypothesis in terms of both representation and meaning due to basing its operation on the statement of the theorem alone when she needed to complete the 3rd theorem by determining the hypothesis and judgment, she could advance the proof to a certain level by correcting her mistakes when she decided to understand and interpret the key ideas. She could form the hypothesis and the clause, albeit incompletely, by establishing relations between the key idea '$ab^{-1} \in H_i$ from the definition of the subgroup' and the key idea '$ab^{-1} \in H$ from the definition of intersection'. The incompleteness was her inability to provide an explanation on what H_is was, and her inability to write $\bigcap_i H_i = H$ instead of $\bigcap_{i \in I} H_i = H$ due to her lack of mathematical notation knowledge. She expressed that the hypothesis and the clause could be determined just from '$ab^{-1} \in H$ from the definition of intersection' even if '$ab^{-1} \in H_i$

from the definition of subgroup' were not provided. From this statement, it is thought that she considers H_is as subgroups but she did not express in writing.

Findings Regarding the 4th Theorem

In the proof of the 4th Theorem, where certain inferences were expected, the prospective teachers made certain inferences using the prime number definition given as a key idea, and dealing with the information given as key ideas again with a holistic approach, and then tried to complete the proof with those inferences. Although the prospective teachers made the expected inference and could heuristically understand the proof by using the definition of prime number in this theorem, all of them except Gul had difficulty expressing it in writing. Ceyda, on the other hand, stated that she actually knew Lagrange's theorem, but the idea that she needed to use it in this proof arose with the help of the key idea.

Presented below are findings and comments regarding Umut's proof completion process as an example of the analysis of Theorem 4, in which prospective teachers' proof completion processes were examined with the help of key ideas.

Theorem.4: Every group G with prime order is cyclic.

Proof.4: Let us suppose that $o(G) = k$ and k is a prime. We show that G is a cyclic group.

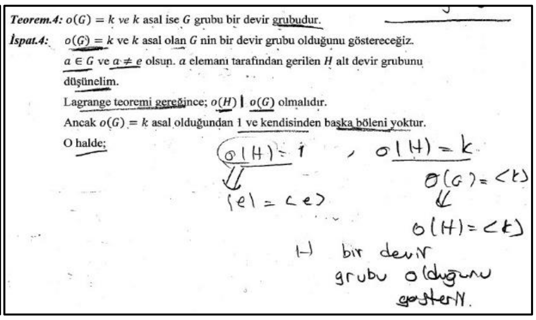
Take $a \in G$ which is different from the identity of G. We consider the subgroup H that is cyclic and generated by a.

Thus we have $o(H) \mid o(G)$ by the Lagrange theorem.

But $o(G) = k$ and is prime, it has just only divisors which are 1 and itself.

We see that $o(H) = k$. Therefore the group G is the same as the group H. That is $H = (a)$ and $o(H) = k$. As a result

$G = (a) = H$.



Teorem.4: $o(G) = k$ ve k asal ise G grubu bir devir grubudur.

İspat.4: $o(G) = k$ ve k asal olan G nin bir devir grubu olduğunu göstereceğiz.

$a \in G$ ve $a \neq e$ olsun. a elemanı tarafından gerilen H alt devir grubunu düşünelim.

Lagrange teoremi gereğince; $o(H) \mid o(G)$ olmalıdır.

Ancak $o(G) = k$ asal olduğundan 1 ve kendisinden başka böleni yoktur.

O halde;

Figure 4. Proof of the 4th Theorem Given in the Proof Completion Form and Umut's Solution

In the proof of Theorem 4, which was expected to be completed using the given key ideas and the description of prime numbers, Umut made some inferences by utilizing both the given key ideas and the description of prime numbers. However, he could not complete the proof since he could not support his inferences through key ideas, and therefore could not exhibit a holistic approach. In the proof, where an interpretation like 'If the subgroup is a cyclic group, G is a cyclic group' was expected, he produced an incorrect thought by saying 'if G is a cyclic group, I say the subgroup H is a cyclic group as well'. Umut dealt with the order of the group and the element producing the group with the same presentation and failed to express his thoughts in accordance with the mathematical notation. The fact that Umut had previously written '$o(H) = k$' correctly but then wrote it incorrectly as '$o(H) = \langle k \rangle$' was thought to result from his carelessness and untidiness, not from his lack of mathematical notation knowledge.

Findings Regarding the 5th Theorem

In the 5th Theorem, the prospective teachers expected to remember the homomorphism property were asked to do operations by using the associative property and identity element property, and they were given equality as a key idea, which was the starting step of the operation. All the prospective teachers advanced their operations based on the key idea. At this point, Umut eliminated his mistakes by correlating the result he found with the key ideas provided before and after the process steps.

Presented below are findings and comments regarding Ezgi's proof completion process as an example of the analysis of Theorem 5, in which prospective teachers' proof completion processes were examined with the help of key ideas.

Theorem.5: Let G be a group and $a \in G$ Then the map $\varphi_a: G \to G$ defined by

$$\boldsymbol{\varphi_a(x) = axa^{-1}}$$ is an inner automorphism of G for each $x \in G$.

Proof.5: It is easy to see that φ_a is a homomorphism so that

$$\boldsymbol{\varphi_a(xy) = a(xy)a^{-1} = a(xa^{-1}ay)a^{-1} = (axa^{-1})(aya^{-1})}$$
$$= \boldsymbol{\varphi_a(x)\varphi_a(y)}$$

for each $x, y \in G$. Furthermore,

$$\varphi_a(x) = \varphi_a(y) \Longrightarrow axa^{-1} = aya^{-1} \Longrightarrow x = y$$

i.e. φ_a is one to one. In addition there exist $x \in G$ such that

$$\varphi_a(x) = axa^{-1} = y$$

for each $y \in G$ i.e. φ_a is onto. As a result φ_a is an automorphism of G.

Teorem.5: G bir grup ve $a \in G$ olsun. $\forall x \in G$ için,

$$\varphi_a(x) = axa^{-1}$$

ile tanımlı $\varphi_a: G \to G$, G nin bir otomorfizmasıdır. φ_a ya G nin bir iç otomorfizması denir.

İspat.5: $\forall x, y \in G$ için;

$$\varphi_a(xy) = a(xy)a^{-1} = axeya^{-1}$$
$$= axa^{-1}aya^{-1}$$
$$= \varphi_a(x).\varphi_a(y)$$

axa^{-1} aya^{-1}

Olduğundan, φ_a bir homomorfizmadır.

$$\varphi_a(x) = \varphi_a(y) \Longrightarrow axa^{-1} = aya^{-1} \Longrightarrow x = y$$

Olduğundan φ_a 1-1 dir.

$\forall y \in G$ için $\varphi_a(x) = axa^{-1} = y$ olacak şekilde $\exists x \in G$ bulunabileceğinden, φ_a örten de olur. Şu halde φ_a G nin bir otomorfizmasıdır.

Figure 5. Proof of the 5th Theorem Given in the Proof Completion Form and Ezgi's Solution

Ezgi started with operations that would form the proof using the key ideas of "It is easy to see that φ_a is a homomorphism" and "$\varphi_a(xy) = a(xy)a^{-1}$" given in the proof of Theorem 5 and "$\varphi_a(x) = axa^{-1}$" provided in the theorem statement. At this point, she was able to apply the identity element property to her operations by using it in accordance with its purpose, but she could not complete the proof exactly due to omitting to define and use the associative property.

Discussion and Conclusion

In this study, an application consisting of proofs of five theorems supported by the key ideas was used to reveal how prospective teachers can use the key ideas. Research on proof requires dealing with complexity (Reid, 1999). The proof process is divided into parts so as to relatively eliminate this complexity and reveal the proof processes of individuals in detail. In practice, some sections were left incomplete, and hints referred to as key ideas were made available to complete the section left incomplete. Since difficulties may be faced while coordinating and using all information in the proof process (Moore, 1990), key ideas in this implementation are given in a particular order in the proof , and by doing so, it is tried to identify the states of prospective teachers regarding their ability to use the given key ideas to be able to complete the sections left incomplete.

Key ideas make it easier for prospective teachers to comprehend the proof intuitively, develop a positive attitude towards the proof, start proving, do their algebraic operations, and complete the proof process. The prospective teachers corrected the incorrect steps that they had taken during the proving process, with the help of the key ideas again. Supporting theorems with key ideas, also affected the performance of the participant prospective teachers in proving the theorem; however, the conceptual knowledge or competence levels of the prospective teachers related to the proof of the theorem affected the level of benefiting from key ideas as well (for example; although Ceyda used the key ideas, she failed in completing the proof because she perceived the operations as a distributive property, based on the definition of the normal subgroup). These results obtained through the research show consistency with the findings obtained through the research of Karaoglu (2010).

Individuals do not know what information must be included at what stage of the proof. In a way supporting our results, Moore (1990) states that individuals have difficulty coordinating and using all the information in a proven theorem at the same time. In addition, according to Agustyaningrum et al. (2020), individuals experience problems such as not knowing how to use definitions, axioms, or theorems in the process of proving, and not being able to intuitively understand the concepts they need. Key ideas that enable understanding and detailing (Raman, 2002) give individuals the opportunity to know at what stage of proof knowledge is used or how to coordinate and use their knowledge. Also in this research, key ideas were observed to play an important role in intuitively understanding the proof. Therefore, prospective teachers do not have difficulty completing a proof supported by key ideas. Nevertheless, those who have basic incompetence and had learned incorrectly cannot use key ideas, no matter how easy it is to facilitate a proof with the help of key ideas. However, key ideas also here provide the opportunity to reveal students' deeper incorrect knowledge. To be able to complete a given proof supported by key ideas, an individual must also know what information he/she needs to use, and accordingly, due to information deficiencies, he/she cannot complete the proof. Based on all these, key ideas can be said to play an active role in the proving process and teaching proof.

According to Selden and Selden, (2008), teachers should not teach a complex process such as proofs only through lectures and by trying to convey existing proofs directly to the student; they should also take into consideration the mutual interactions in the form of teacher-student and student-student. Giving students helpful tools that will facilitate their intuitive understanding of proofs can make complex ideas more comprehensible by helping students with reasoning and may increase students' participation in the proving process (Gallagher & Infante, 2021). Key points and ideas positively affect the performance of students in the proof process (Karaoglu, 2010). Devoting a central place to key ideas in the high school and university curricula seems to be an important step for helping students develop a positive perspective on mathematical proofs (Raman, 2003). Therefore, getting help from key ideas in the teaching process and including proofs supported by key ideas in the lecturing process are thought to contribute to proof teaching. If students are taught how to prove a theorem with the help of key ideas, they can be enabled to determine the basic points of the proof at the end of this teaching process and intellectually shape the hints that can help them in creating the proof from the beginning. Therefore, they will be able to prove it without the need for memorization.Considering the fact that prospective mathematics teachers learn hundreds of theorems and their proofs during their education, the learned information should be prevented from being acquired based on rote-learning, by developing activities that would enable prospective teachers to internalize proving. (Guler, Ozdemir & Dikici, 2012). Developing activities consisting of proofs supported by key ideas can contribute to the prospective teachers' ability to prove without memorization and internalize proving.

This research is limited to theorem proofs achieved through the direct proof method only, and the theorems intended for determining the proof method are not included in the research due to the intensity of the application; so, this situation was excluded but expected to be discussed in a different research. Revealing the sources of the difficulties that students have in determining the method, through similar research where students are asked to complete the steps of determining the proof method by providing them with proofs supported by key ideas, on the condition that the given key ideas differ in each question, may be the subject of a different research.

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